

Geometric quantization of generalized oscillator

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Abstract

Using geometric quantization procedure, the quantization of algebra of observables for physical system with Ricci-flat phase space is obtained. In the classical case the appointed physical system is reduced to harmonic oscillator when the one real parameter is vanished.

Keywords: Generalized oscillator, Geometric quantization, Ricci-flat Kähler manifold.

1. Introduction

There exist a number of procedures and methods to quantize a physical systems, but it is geometric quantization that takes into account the geometrical background (i. e. geometry of phase space) of the physical system. The procedure of geometric quantization was discovered by B. Kostant [2] and J.M. Souriau [4] in 1970. During last 25 years it was highly developed by multiply authors. J.M. Tuynman in [5] has compared some known methods of quantization (in particular, geometric quantization) using 2-dimensional generalized harmonic oscillator, i. e. hamiltonian system with algebra of observables $\mathfrak{su}(1,1)$. For physical

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applications it would be helpful to quantize the generalized harmonic oscillator for an arbitrary dimension. Theorem 3 of the present paper gives such quantization for even dimensions.

Recently a Ricci-flat Kähler metric for any real dimension $4n$ was constructed by the author [6]. The metric has the following form

$$g_{\alpha\bar{\beta}} = u'' \overline{z^\alpha} z^\beta + u' \delta_\beta^\alpha = \frac{a^m}{r^2} (r^m - a^m)^{\frac{1-m}{m}} \overline{z^\alpha} z^\beta + \frac{(r^m - a^m)^{\frac{1}{m}}}{r} \delta_\beta^\alpha, \quad (1)$$

where

$$u' \equiv \frac{du}{dr} = \frac{(r^m - a^m)^{1/m}}{r}, \quad (2)$$

$u'' \equiv du'/dr$, $r \equiv \sum_{\alpha=1}^m z^\alpha \overline{z^\alpha}$, $m \equiv 2n$, $\mathbf{R} \ni a = \text{const.}$ As it was shown in [6] (this proposition is almost evident) the group of complex isometries of the metric (1) is $\text{SU}(m)$. It is notable that Ricci-flat Kähler phase space M with abovementioned metric has an subalgebra \mathcal{F} of Poisson algebra of $C^\infty(M)$ -functions on M and \mathcal{F} is appropriate to quantize by geometric quantization procedure. Let us call the physical system with phase space M as *generalized oscillator* and the algebra \mathcal{F} as *algebra of observables of generalized oscillator*.

2. Antiholomorphic polarization

Let us consider a Kähler polarization (see [2, 4] for details) $F \subset TM \otimes_{\mathbf{R}} \mathbf{C}$ which is determined by the condition

$$F_p = \{X \in TM_p \otimes \mathbf{C} \mid X = \xi_{\alpha|p} \text{ad}(z^\alpha)_p, \xi_{\alpha|p} \in \mathbf{C}\}, \quad (3)$$

where $\xi_{\alpha|p}$ are $C^\infty(M)$ -functions of a point $p \in M$. The defined polarization is called *antiholomorphic polarization*. Function $f \in C^\infty(M)$ preserves polarization F if it obeys to the

equation

$$L_{\text{ad } (f)} X \in F \quad \text{for all } X \in F. \quad (4)$$

It is equal to the next condition

$$[\text{ad } (f), \text{ad } (z^\mu)] = \overset{(f)}{a}{}^\mu{}_\beta \text{ad } (z^\beta). \quad (5)$$

Theorem 1 [1] *Let M be a Kähler manifold and $F \subset TM \otimes_{\mathbf{R}} \mathbf{C}$ be an antiholomorphic polarization. Function $f \in C^\infty(M)$ preserves polarization F if and only if in every complex chart $(U, z^\alpha, \bar{z}^\alpha)$, $\alpha = 1, \dots, m$, on M the equation*

$$f = \partial_\alpha \Phi \varphi_\alpha(z) + \chi(z), \quad (6)$$

holds. Here $\varphi^\alpha(z), \xi(z)$ are an arbitrary holomorphic functions.

On the manifold (M, g) where g is defined by (1) the formula (6) takes the following form

$$f = \sum_\alpha u' \bar{z}^\alpha \varphi_\alpha(z) + \chi(z), \quad (7)$$

where u' is defined by (2).

Let us consider the next functions

$$N^{\alpha\bar{\beta}} = u' z^\alpha \bar{z}^\beta, \quad \alpha, \beta = 1, \dots, m. \quad (8)$$

It is easy to show that the fuctions $N^{\alpha\bar{\beta}}$ preserve polarization F . From here and from theorem 1 we find

$$N^{\alpha\bar{\beta}} = u' \bar{z}^\sigma \varphi_\sigma^{\alpha\bar{\beta}}, \quad (9)$$

where $\varphi_\sigma^{\alpha\bar{\beta}} = z^\alpha \delta_\sigma^\beta$.

The Hamiltonian vector fields of functions $N^{\alpha\bar{\beta}}$ are defined by the following equalities

$$V^{\alpha\bar{\beta}} \equiv \text{ad} (N^{\alpha\bar{\beta}}) = i(z^\alpha \partial_\beta - \bar{z}^\beta \partial_{\bar{\alpha}}).$$

The Poisson brackets of functions $N^{\alpha\bar{\beta}}$ and the commutators of its Hamiltonian vector fields have the next form

$$\{N^{\alpha\bar{\beta}}, N^{\mu\bar{\nu}}\} = i(\delta_\mu^\beta N^{\alpha\bar{\nu}} - \delta_\nu^\alpha N^{\mu\bar{\beta}})$$

$$[V^{\alpha\bar{\beta}}, V^{\mu\bar{\nu}}] = \delta_\nu^\alpha V^{\mu\bar{\beta}} - \delta_\beta^\mu V^{\alpha\bar{\nu}},$$

Since $V^{\alpha\bar{\beta}}$ are Hamiltonian and holomorphic that the corresponding transformations preserve complex structure on M and fundamental 2-form $\Omega(X, Y) \equiv g(JX, Y)$. Hence they preserve metric g and they are Killing vector fields on M which preserve complex structure. So the vector fields $V^{\alpha\bar{\beta}}$ form the algebra $\mathfrak{su}(m)$ of infinitesimal holomorphic isometries of the metric (1).

3. Geometric quantization of the algebra of observables

The transformations from group $SU(m)$ preserves complex structure and metric g as well as fundamental form Ω . This means that the action of $SU(m)$ on M is symplectic [3]. The cohomology group $H^2(\mathfrak{su}(m), \mathbf{C})$ is trivial and as it was shown in [3] the action of $SU(m)$ on M is Poisson action.

Let us consider an algebra (with respect to Poisson brackets) $\mathcal{F}(m)$ of linear functions on $N^{\alpha\bar{\beta}}$, $\alpha, \beta = 1, \dots, m$. As it was mentioned in Introduction, this algebra is called as algebra of observables of $2m$ -dimensional generalized oscillator. It is evident that $\mathcal{F}(m)$ coincides with an algebra of functions preserving antiholomorphic polarization on M . For such a functions the following theorem exists.

Theorem 2 [1] *Let M be a Kähler manifold, $F \subset TM \otimes_{\mathbf{R}} \mathbf{C}$ be an antiholomorphic polarization and $\mathcal{F}_F(M) \subset C^\infty(M)$ be an algebra of functions on M which preserve polarization F . Then in every chart $(U, z^\alpha, \bar{z}^\alpha)$, $\alpha = 1, \dots, m$, the quantization \mathcal{Q} of $\mathcal{F}_F(M)$ is defined by the formulae*

$$\mathcal{Q}(f)\psi \cdot \mu_0 = (\chi + \hbar(\varphi_\sigma \partial_\sigma + \frac{1}{2} \partial_\sigma \varphi_\sigma))\psi \cdot \mu_0, \quad (10)$$

where $f = \varphi_\tau \partial_\tau \Phi + \chi \in \mathcal{F}_F$, χ, ψ is holomorphic functions on U and μ_0 is non-vanished at every point of U section of the Hermitian vector bundle \mathcal{L} over M .

One can construct the quantization of algebra $\mathcal{F}(m)$ using theorem 2. As far as the map \mathcal{Q} is linear, it is sufficient to define the action of \mathcal{Q} on functions $N^{\alpha\bar{\beta}}$.

Using (9) we find from (10)

$$(\mathcal{Q}(N^{\alpha\bar{\beta}}))\psi \cdot \mu_0 = \hbar \left(z^\alpha \frac{\partial \psi}{\partial z^\beta} + \frac{1}{2} \delta_\beta^\alpha \psi \right) \cdot \mu_0. \quad (11)$$

The last formula defines quantization of algebra $\mathcal{F}(m)$ of observables of m -dimensional generalized oscillator.

By summation (11) with $\alpha = \beta$ from 1 to m we have

$$(\mathcal{Q}(H))\psi \cdot \mu_0 = \hbar \left(z^\sigma \frac{\partial \psi}{\partial z^\sigma} + \frac{m}{2} \psi \right) \cdot \mu_0,$$

Therefore, the eigenvalues of operator $\mathcal{Q}H$ are defined by the next relation

$$\lambda_l = \hbar \left(l + \frac{m}{2} \right), \quad l = 0, 1, 2, \dots$$

and coincide with energy levels of harmonic oscillator in flat space.

4. Corollary

The main result of the paper can be formulated as the following

Theorem 3 *Let (M, g) be Kähler Ricci-flat space, $\dim_{\mathbf{R}} M = 2m = 4n$ with metric g , defined by (1). Let F be an antiholomorphic polarization defined by (3) and $\mathcal{F} \subset C^\infty(M)$ be an algebra of functions on M which are linear functions on variables $N^{\alpha\bar{\beta}}$ defined by (8). Then in every chart $(U, z^\alpha, \bar{z}^\alpha)$, $\alpha = 1, \dots, m$, the quantization \mathcal{Q} of \mathcal{F} is defined by operators (11) where ψ is holomorphic function on U and μ_0 is non-vanished on U section of Hermitian vector bundle \mathcal{L} over M .*

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